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2002 J. Phys.: Condens. Matter 14 2379

(http://iopscience.iop.org/0953-8984/14/9/327)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS: CONDENSED MATTER

J. Phys.: Condens. Matter **14** (2002) 2379–2390 PII: S0953-8984(02)31189-5

Random multiplicative processes and the response functions of granular packings

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Received 23 November 2001
Published 22 February 2002
Online at stacks.iop.org/JPhysCM/14/2379

Abstract

It has recently been suggested that the property of isostaticity of the contact network of a frictionless polydisperse granular packing in the limit of low applied pressure is responsible for some of the anomalous static behaviour of packings. In this paper we discuss the fact that, on disordered isostatic networks, displacement–displacement and stress–stress static Green functions are described by coupled random multiplicative processes and thus have a truncated power-law distribution, with a cut-off that grows exponentially with distance. The expectation values of Green functions on these systems differ from observed averages by an exponentially large factor unless the number of samples over which averages are taken is exponentially large. Thus predicted averages will seldom be observed in experiments. If the external pressure is increased sufficiently, excess contacts are created, the packing becomes hyperstatic, and the above-mentioned anomalous properties disappear because Green functions now have a bounded distribution. Thus the low-pressure, isostatic, limit is a *critical point* where the Green function distribution becomes scale-free. This criticality is induced by multiplicative noise.

1. Introduction

In this paper, the static properties of granular packings are discussed with the aid of rigidity concepts recently put forward [1–3]. Here, we concentrate mainly on response functions and their properties on isostatic networks. Disordered packings of frictionless, cohesionless spheres have, in any dimension d, an *isostatic* contact network when the external pressure is small with respect to the stiffness of its particles. Because of isostaticity, the stress induced in bond b by a vertical load applied at site i, is exactly equal to the vertical displacement Δ_b^i that site i suffers when the interparticle bond b is stretched. This symmetry between force–force and displacement–displacement response functions links together two apparently unrelated phenomena observed in granular systems: (a) stress distributions which are not Gaussian as in 'usual' disordered media, but much wider, perhaps with an exponential or stretched-exponential tail [4–14]; and (b) large-scale rearrangements under small perturbations [9,13,15]. Both phenomena are due to an 'anomalous' probability distribution of Green functions on isostatic systems.

In this paper, it is argued that on generic disordered isostatic networks, Green functions are the result of a multi-dimensional *random multiplicative process* [16], and therefore it is expected that they display power-law distributions. Several examples of linear stochastic processes with multiplicative noise have been studied recently and found to give rise to power-law distributions [17–21]. With the help of two-dimensional numerical models of isostatic packings, we show that Green functions have a truncated power-law distribution, with a cut-off that grows exponentially with distance. These two characteristics are the signatures of random multiplicative processes. The slow convergence of averages towards their expectation values (EVs) is another [16], and has the consequence that averages differ radically from EVs unless the number of disorder realizations is huge. This is confirmed numerically for two-dimensional model isostatic networks and is very relevant for the interpretation of recent experimental work [22, 23].

When the packing is compressed, the resulting network is no longer isostatic and the observed abnormal properties of granular materials give place to a 'normal' elastic behaviour, i.e. a bounded distribution of Green functions. Therefore, disordered networks in the isostatic limit show a 'noise-induced' transition whereby the distribution of response functions acquires a critical (power-law) character.

2. Green functions are random multiplicative variables on disordered isostatic networks

On isostatic networks, the stress λ_{mn} induced in a contact (mn) by a unit force acting on site i in the direction $\vec{e}_\alpha$, is equal to the component in the direction $\vec{e}_\alpha$ of the displacement induced in site i by a unit stretching of bond mn [1–3, 24, 25]. Thus, force–force and displacement–displacement response functions are equal on isostatic systems (but notice that they are defined in such a way that propagation of forces and displacements occurs in opposite directions). These remarks only apply for *infinitesimal* perturbations. Our considerations do not hold for response functions when rearrangements of the contact network are allowed [24].

Let us first consider a square lattice (figure 1(a)). Because of the above-mentioned symmetry in response functions, the vertical stress Green function G_y is obtained by measuring the vertical displacement Δ_y induced on a given site by a vertical displacement of a site below it. If our system is originally isostatic (minimally rigid), this displacement produces a deformation without any change in bond lengths. Only those sites along the diagonals (characteristics) stemming from the perturbation point will be displaced. Consequently, G is some constant on those sites along the diagonals, and zero everywhere else.

We now introduce positional disorder by randomly displacing the sites of the square lattice by a small amount, as shown in figure 1(b). Because the bond angles are now random, some of the displacement diffuses inside the cone defined by the two characteristics. One notable feature is that the displacement of some sites inside this cone may now be negative, meaning that an overweight there would reduce the stress on the bottom site. Moreover, it is easy to see that the propagation of displacements δ is a *random multiplicative process*, as we now show. Considering first propagation along one of the characteristics and given that the sites below the cone (shaded sites in figure 1(b)) do not move, this is equivalent to propagation of displacements along a one-dimensional chain of sites as depicted in figure 1(c). Let $\vec{\mu}_i$ be the unit vector normal to the bond supporting site i. Since displacements preserve bond lengths, site i can only move along the direction of $\vec{\mu}_i$. Let δ_i be the amplitude of its displacement. Analogously site

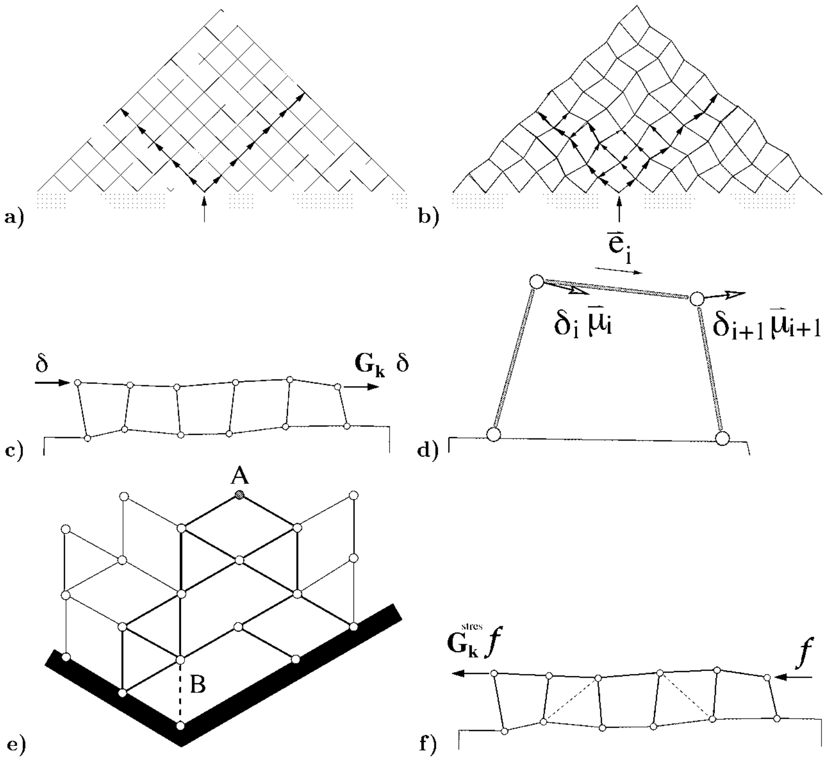

Figure 1. (a) The response function of a regular square lattice is nonzero only along the diagonals stemming from the perturbation point. (b) If positional disorder is introduced, displacements diffuse inside the cone and propagate multiplicatively along any path. (c) Propagation of displacements along the diagonals. (d) Site motions preserve bond lengths, thus the propagation of displacement is a random multiplicative process (see text). (e) A pantograph: when bond B (dashed) is stretched by an infinitesimal amount δ, site A (shaded) moves upwards by an amount 2δ. (f) The propagation of forces and displacements in an overconstrained network is damped, because the system is rigid and opposes deformation by storing elastic energy. Thus the response functions decay with distance.

$i+1$ moves by an amount $\delta_{i+1}\vec{\mu}_{i+1}$. Now let $\vec{e}_i$ be the unit vector from i to $i+1$. The condition that the length of the 'horizontal' bond be unchanged reads $(\delta_i\vec{\mu}_i - \delta_{i+1}\vec{\mu}_{i+1})\cdot\vec{e}_i = 0$, from which we get the recursion relation $\delta_{i+1} = \delta_i(\vec{\mu}_i\cdot\vec{e}_i)/(\vec{\mu}_{i+1}\cdot\vec{e}_i)$. Therefore after k steps, $\delta_k = G_k\delta_0$ with

$$G_k = \prod_{i=0}^{k-1} \frac{\vec{\mu}_i\cdot\vec{e}_i}{\vec{\mu}_{i+1}\cdot\vec{e}_i} \tag{1}$$

showing that G is the result of a random multiplicative process [16] along the two characteristics. The multiplier $(\vec{\mu}_i\cdot\vec{e}_i)/(\vec{\mu}_{i+1}\cdot\vec{e}_i)$ can take values smaller and larger than one. Therefore, values of G_k much larger and much smaller than one will appear for large k. Notice however that, because of the left–right symmetry of the problem, after disorder average one must have $\langle G_k\rangle = 1$, but for each realization of disorder G_k may take very large (of order $\mathrm{e}^{c_1 k}$) and very small (of order $\mathrm{e}^{-c_2 k}$) positive values. In other words the distribution $P(G_k)$ at each site is such that its first moment is one, but higher moments grow exponentially with distance [16].

Next we consider any path inside the cone whose vertex is the displaced site. We assume for a moment that the 'supporting sites' (sites not in the path but connected to it by an upwards-pointing bond) did not move. Under this approximation, propagation of displacements along any path would be a random multiplicative process of exactly the same kind as described by equation (1). The contribution of the motions of the supporting sites can be thought of in a first approximation as 'additive noise' on top of a multiplicative process. It is known that random multiplicative processes with additive noise give rise to truncated power-law distributions [17–21]. A more accurate description for the propagation of displacements would be given by a $(d-1)$-dimensional set of multiplicative variables with local coupling. These systems also display power-law distributions.

A similar conclusion may be reached for isostatic networks with contact disorder. We may now think of a network whose sites are located on a regular triangular lattice, but its bonds are an isostatic subset of the triangular lattice [1–3]. Considering now the propagation of displacements along arbitrary paths on such networks, we again find that they can be roughly though of as a random multiplicative process with additive noise. On contact-disordered lattices, the multipliers take discrete values which include zero and negative values, instead of being continuously distributed as in the previous example. The multiplication of displacements is in this case due to the existence of particular configurations of contacts (called 'pantographs'—figure 1(e)) which multiply displacements by an integer number.

These considerations suffice to remark that the existence of random multiplicative processes in the propagation of stresses and displacements is a generic property of isostatic disordered networks. Numerical results described in section 3 show that Green functions have an approximately symmetric, truncated power-law distribution, with a cut-off that grows exponentially with distance, on isostatic networks.

Strongly compressed granular packings have a hyperstatic contact network, with more contacts than needed for minimal rigidity. The equivalence between stress–stress and displacement–displacement Green functions is no longer valid, thus one must distinguish between G^{stress} and G^{displ}. Moreover, on hyperstatic networks, displacements and stresses cannot be calculated by local propagation. However we can still argue in simple terms that the presence of 'overconstraints' removes the anomalous properties of $P(G)$. For this purpose we consider, for example, the one-dimensional propagation of displacements or forces along one of the characteristics on the distorted square lattice with extra bonds, as shown in figure 1(f). Because this system is hyperstatic, excess constraints make it rigid: its bonds store elastic energy when the network is deformed, and this introduces a dissipative term in the equations that rule the propagation of δ_k [25]. Therefore the average perturbation (and all its higher moments) decay with distance. In this sense, overconstraints introduce a sort of 'damping term' in the (discrete) equations which rule the propagation of perturbations along the chain. A similar mechanism operates on d-dimensional hyperstatic networks.

3. Numerical results

We first consider a two-dimensional triangular packing of $N = L^2$ slightly polydisperse discs of radius $\approx R$ [1–3, 25]. All particles have a unit weight. Boundary conditions are periodic in the horizontal direction. The width of the system is twice its height, therefore force lines stemming from one particle do not 'interfere' with each other around the packing. Each particle has three lower neighbours on which it can eventually 'rest', i.e. establish a contact with. The total number of contacts will in general depend on the 'applied pressure', being $2N$ for low pressure and $3N$ for large enough pressure. The applied pressure is not an explicit parameter in our model, we only mimic its effect by controlling the density O_{v} of three-legged sites in the contact network.

The pile is built by adding layers *from top to bottom*. All particles in a layer have the same height. For each particle, we choose one among the three possible configurations of two

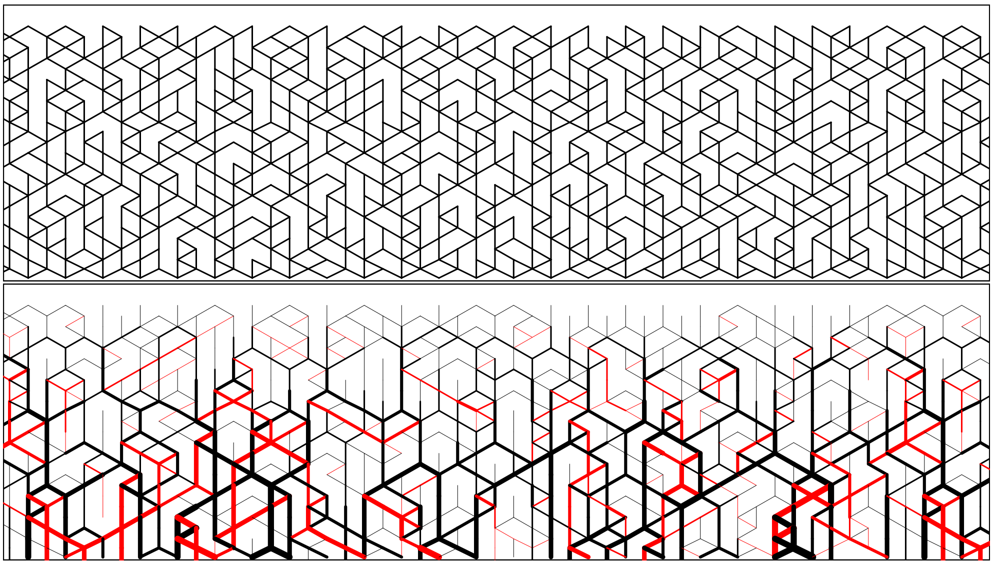

Figure 2. An isostatic packing built by choosing configurations L, S and R entirely at random (RIM model), with $p_L = p_S = p_R = 1/3$. The upper drawing shows the configuration of contacts, and the lower drawing shows stresses on them, with line thicknesses proportional to $\log(|\lambda|)$ (black lines are compressive, dashed lines are tensile stresses, unstressed bonds are not shown).

(This figure is in colour only in the electronic version)

supporting contacts, respectively labelled L (leftwards bond plus vertical bond), S (symmetric), and R (rightwards plus vertical). The type of contact disorder is determined by a triplet of numbers $\{p_L, p_S, p_R\}$ adding up to one, which give the *a priori* probability to choose configurations L, S and R, respectively. The case $p_S = 1$ is the square lattice. For $p_S = 0$ we have that stresses propagate only vertically along independent columns, and diagonal bonds suffer zero stress (because we only have vertical gravitational forces). In both these limits there is effectively no contact disorder for the loading conditions described here. The preferred choices in this paper are usually $p_L = p_S = p_R = 1/3$; an example is displayed in figure 2.

The procedure described above produces isostatic networks, but has *no built-in sign constraint* for stresses, and therefore tensile stresses appear. We call it the 'random-stress' isostatic model (RIM). It is easy to introduce a modification in order to only obtain compressive stresses, as follows: (a) If configuration S is chosen at a given site (which happens with probability p_S), and since the 'incoming forces' are also compressive by construction, we can rest assured that the supporting stresses will also be compressive. (b) If configuration S is not chosen (which happens with probability $1 - p_S$), then we look at the horizontal component of the incoming forces. If it points leftwards then we choose configuration L. If it points rightwards we choose configuration R, and if it is exactly zero we choose any of those. We call this the compressive-stress isostatic model (CIM). An example is shown in figure 3. Once the supporting configuration (L, S or R) of a particle is chosen, we can calculate stresses on both supporting legs, because we already know the 'incoming' forces due to contacts with the previous (upper) layer. Thus, when the bottom is reached and the pile is finally built, all stresses are already known. Green functions are then calculated by upwards propagation in a similar fashion.

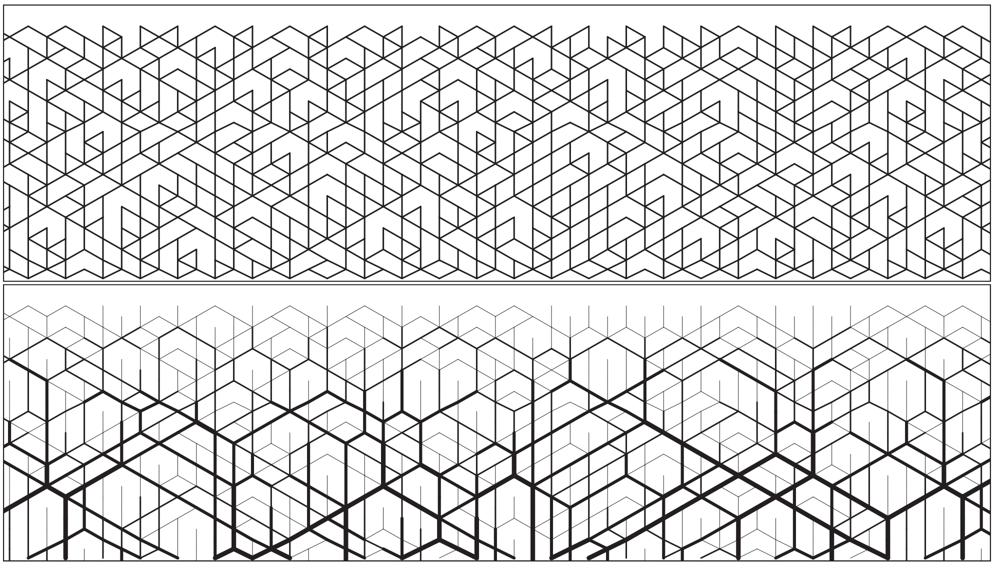

Figure 3. A CIM network with $p_L = p_S = p_R = 1/3$. The upper drawing shows the configuration of contacts, and the lower drawing shows stresses on them, with line thicknesses proportional to $\log(|\lambda|)$. The contact configurations are chosen so as to ensure positive (compressive) stresses on all contacts.

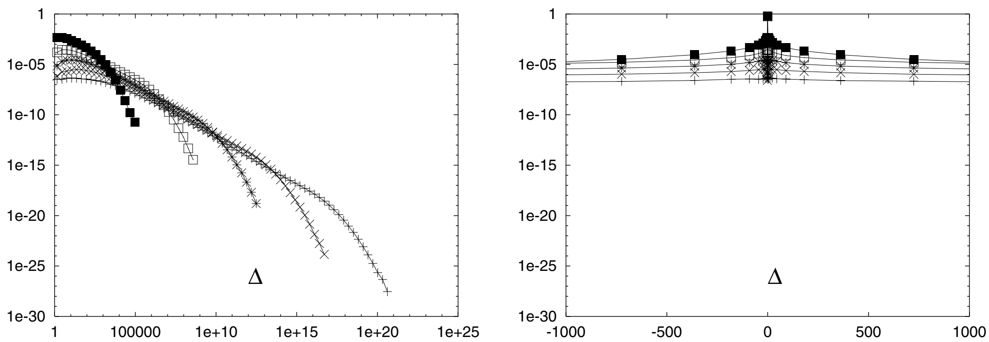

Figure 4. CIM: distribution of stress–stress Green functions Δ (the excess stress induced by a localized extra weight on the top) in log–log (left) and log–lin (right) scales, measured at a depth h of 200 (filled squares), 400 (squares), 600 (asterisks), 800 (crosses) and 1000 (pluses) layers. The lines are drawn only for guiding the eye. Averages were done over 10^4–10^5 realizations, each containing 10^6 discs with $p_R = p_L = p_S = 1/3$.

3.1. Green functions

As discussed previously we expect Green functions to display broad distributions with respect to disorder realizations, because they are determined by random multiplicative processes. The numerical results presented next show that this is indeed the case.

Figure 4 shows Green function distributions for CIM networks, averaged over several thousand disorder realizations, at a given depth below the applied force. We can see that their distribution is a truncated power law, with a cut-off that grows exponentially with distance. The exponent of the power law is close to one, therefore all moments of order larger than one grow exponentially fast with height. According to our discussions in section 2, both features of the distribution of Green functions, namely a truncated-power law form and an exponentially growing cut-off, are due to the existence of random multiplicative processes on disordered isostatic networks. It is known that averages of the products of h random variables converge very slowly to their EVs as the number of realizations grows, since their EVs are dominated by exponentially rare events, i.e. events of exponentially large strength but which happen with an exponentially decreasing probability $\sim \exp(-ah)$. It has been stressed that averages taken over a moderately large but not exponentially large number of disorder realizations give results which differ from EVs by an exponentially large factor [16], and only converge to the EV after averaging over an exponentially large number of realizations. In the case of Green functions on isostatic networks one thus expects to be able to measure true EV after averaging over a number of realizations which grows exponentially with h. These expectations are fully confirmed by the numerical results shown in figure 5. For small and moderately large numbers of samples one obtains an average value $\langle G \rangle$ which fluctuates wildly from point to point, and takes positive as well as negative values. After averaging over 10^4 samples, some structure starts to emerge from the noisy background, and finally for numbers of samples of the order of 10^7 the true average, which is non-negative, is obtained. Notice that $\langle G \rangle$ is nonzero mostly along two 'characteristics' stemming from the perturbation point, as predicted by hyperbolic models [26, 27]. However, experiments performed over smaller numbers of disorder realizations will not observe this behaviour. This remark might be relevant for the interpretation of recent experimental work [22, 23].

3.2. Stress distribution

Because of the fact that each particle has a unit gravitational force acting on it, the stress λ_b on a given bond b is given by $\lambda_b = \sum_i \Delta_i^b$. Given that Green functions Δ_i^b have an exponentially broad distribution, one could naively expect stresses to behave in the same way. However, the scale of stresses grows only linearly with depth on CIM models, and not exponentially. It is therefore evident that strong correlations exist, so that Green functions of different signs (and orders of magnitude larger than the stresses themselves) cancel each other exactly at each site. The origin of these correlations is easily discovered. They are a consequence of the fact that for CIM networks we choose the local supporting configurations at each site in order to avoid negative stresses. This is equivalent to requiring that $\sum_i \Delta_i^b$ never be negative, and introduces correlations among Green functions.

When the supporting configurations are chosen entirely at random (RIM) the above-mentioned correlations are absent and stresses of both signs appear (figure 6). The stress distribution $P(\lambda)$ is now approximately symmetric around zero. It is not totally symmetric because its first moment is of course still proportional to the depth as it should be, but $P(\lambda) - P(-\lambda)$ is nonzero only for small stresses. The large-stress behaviour is symmetric. The cut-off in $P(\lambda)$ now *grows exponentially with depth*. The second moment of $P(\lambda)$ grows exponentially with depth (not shown). The fact that the stress-scale is exponentially amplified with distance is a generic property of randomly disordered isostatic structures. As we can see, the sign constraint on stresses, and the resulting correlations induced by it, are crucial to avoid this behaviour.

3.3. Square networks with positional disorder

It has been argued previously in this paper that multiplicative processes are a generic property of disordered isostatic networks. Let us now briefly consider topologically regular square lattices with positional disorder (figure 1(b)). Each disc has a unit weight and rests on two neighbours, but their centres are slightly displaced from the sites of a regular lattice. Our results for Green function distributions and stress distributions are shown in figure 7.

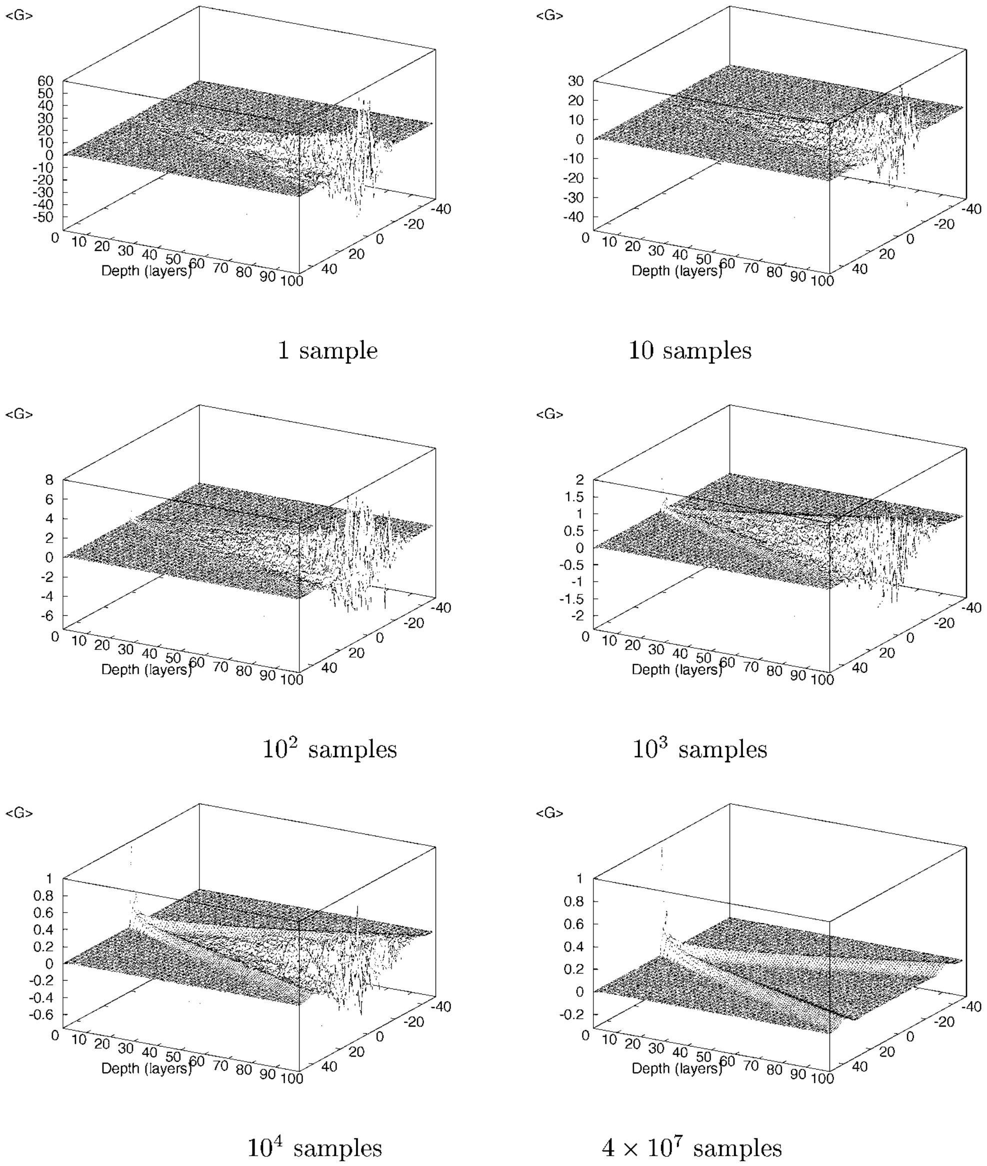

Figure 5. Average response function $\langle G \rangle$, on contact-disordered isostatic networks, for increasing numbers of disorder realizations. For small numbers of samples, exponentially large values of G of both signs appear. These enormous fluctuations cancel out as the number of samples increases and $\langle G \rangle$ turns out to be non-negative everywhere, even when for each given sample G may take exponentially large values of either sign.

The distribution of response functions $P(G)$ at a given depth is very similar to that obtained for contact-disordered lattices (figure 7), i.e. a power-law decay up to an exponentially decreasing cut-off. For square networks, stresses of both signs appear, even when the gravitational forces act downwards. Again the higher moments of the stress distribution are seen to grow exponentially with depth, a result which follows from the exponentially broad distribution of response functions.

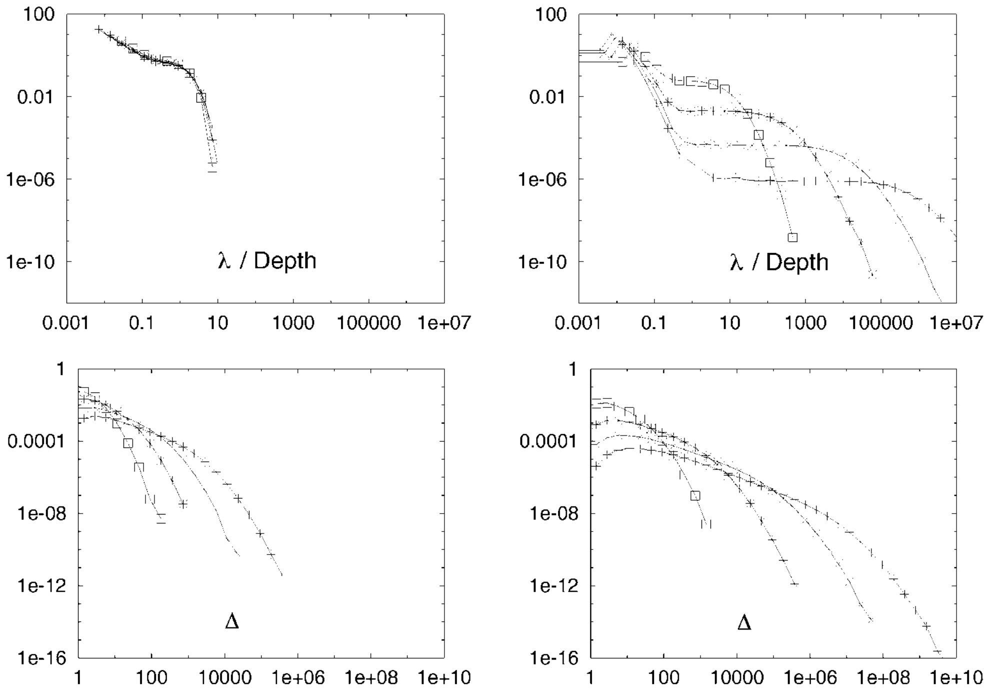

Figure 6. Distribution of normalized stress $w = \lambda/\text{depth}$ (upper plots) and of Green functions Δ (lower plots), for isostatic packings with compressive-only stresses (CIM, left) and unrestricted stresses (RIM, right), at a depth h of 50 (squares), 100 (asterisks), 150 (crosses) and 200 (pluses) layers. Only positive stresses are shown. Notice that for RIM (upper right plot), there is a stress cut-off which grows *exponentially* with depth, instead of being proportional to depth as in the compressive-stress case (upper left).

3.4. The effect of increasing the applied pressure

In order to simulate the effect of increasing the external pressure we introduce a variable density O_v of 'overconstraints'. This is done by letting each site be supported by three legs with probability O_v. This parameter O_v 'mimics' the effect of external pressure. If $O_v > 0$, stresses are no longer determined by local conditions as in the isostatic ($O_v = 0$) case, but depend on the rheology and loading conditions all over the material [1, 2]. In this case the elastic equations must be solved and this is done iteratively by means of a conjugate gradient [28] method, assuming linear elastic bonds. As argued in [1–3] and here, when the pressure applied on a packing is increased, the resulting hyperstatic structures now damp the multiplicative effects, thus eliminating the unusual critical properties of Green functions that isostatic networks display. This is illustrated in figure 8, showing that in the contact-disordered numerical model with $O_v > 0$, the distribution of induced displacements Δ is bounded, and its moments decay with distance instead of increasing exponentially as in the isostatic case. The same conclusion may be drawn from the results in figure 9. One sees that overconstraints induced by external pressure make $P(\lambda)$ become similar to a Gaussian, that is, $P(\lambda) \sim e^{-\lambda^2}$ for large λ. Therefore, the low-pressure limit, in which the system becomes isostatic, can be regarded as a noise-induced critical point. The multiplicative noise effects give rise to power-law distributed Green functions only in the isostatic limit.

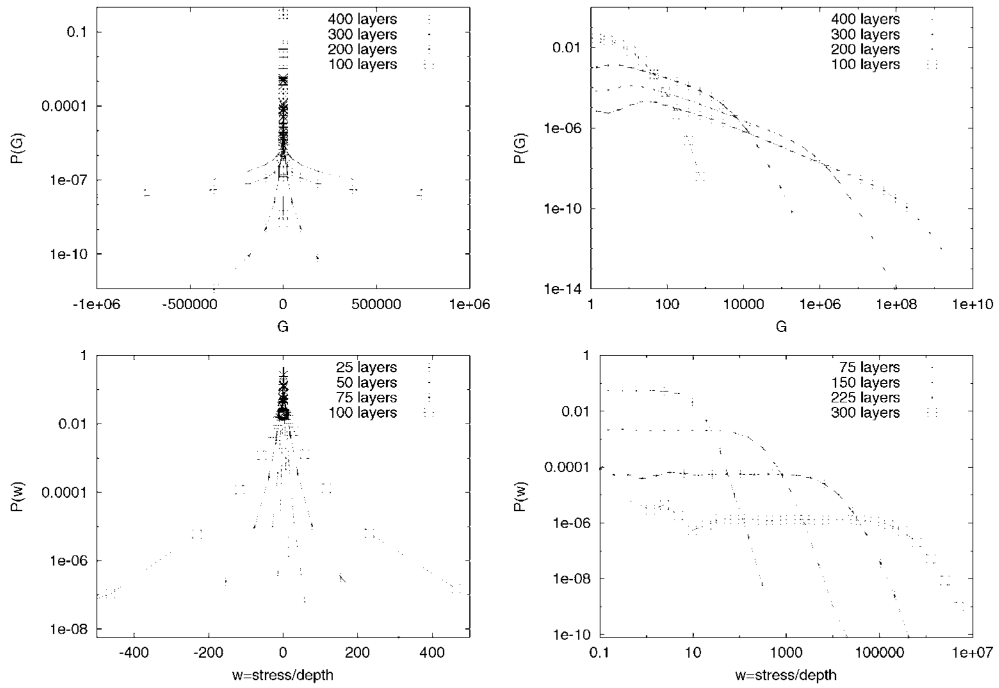

Figure 7. Log–lin (left) and log–log (right) plots of Green function distributions $P(G)$ (upper row) and normalized stress distributions $P(w)$ (lower row) for position-disordered square lattices, measured at several heights or depths.

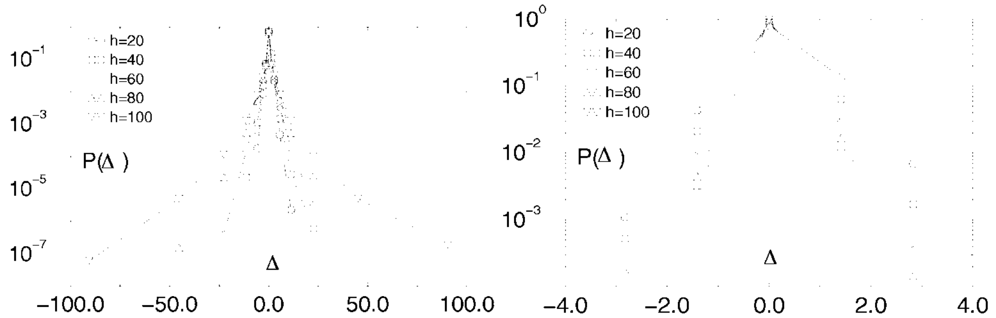

Figure 8. On isostatic lattices (left), $P(\Delta)$, which measures the distribution of load–load and displacement–displacement Green functions, gets broader when the distance h increases. It eventually becomes a truncated power-law distribution (see figure 4) with an exponentially large (of order $e^{\alpha h}$) cut-off. However, a density of overconstraints $O_v = 0.02$ is enough to revert this situation (right). In the overconstrained case, $P(\Delta)$ gets narrower with increasing distance h.

4. Discussion

Rigidity considerations for the contact network of disordered frictionless packings of spheres predict the property of isostaticity in the low-pressure (or large stiffness) limit [1–3]. Isostaticity means that the contact network is rigid and minimally so. The removal of *any* contact would produce the loss of rigidity, and in this sense all contacts are *cutting-bonds* for rigidity percolation [29]. Important consequences of the isostaticity property are as follows. On

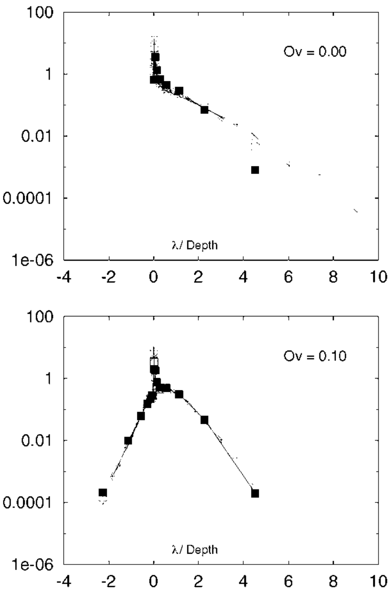

Figure 9. Normalized stress distributions on a CIM model with increasing densities O_v of overconstraints, measured at a depth of 20 (filled squares), 40 (squares), 60 (asterisks), 80 (crosses) and 100 (pluses) layers.

isostatic networks, stress–stress and displacement–displacement Green functions are equal. In the case of sequentially deposited packings we have shown that Green functions Δ of disordered isostatic networks are distributed according to a truncated power law: $P(\Delta, h) \sim \Delta^{-\alpha}$ for $\Delta < \Delta^{\max}(h)$, where the cut-off $\Delta^{\max}(h)$ grows exponentially with distance h. The reason for this surprising behaviour is that on disordered isostatic networks, response functions Δ are the result of random multiplicative processes [16]. Several instances of stochastic processes of this sort have been recently studied and found to give rise [17–21] to power-law distributions.

If only compressive stresses are allowed, nontrivial correlations are introduced among different Green functions at a given point. In this case the resulting stresses, which are the superpositions of Green functions, are 'well behaved', i.e. its nth moment $\langle\lambda^n\rangle$ scales as depth^n. Green functions themselves are however not well behaved and therefore not easily measured experimentally. It is argued in this paper that, because of the existence of random multiplicative processes, average Green functions will only converge to their EVs when the number of samples over which disorder averages are taken is of the order $\exp ah$ where h is the height of the system in layers. We have studied a two-dimensional contact-disordered model of isostatic networks and shown that $\langle G\rangle$ is nowhere negative, and that perturbations propagate mostly along two characteristics, as predicted by hyperbolic models [26, 27]. However, this is only observed when the number of samples is very large. For the moderate numbers of samples within the reach of present experimental techniques [22, 23], averages differ strongly from EVs.

Although granular packings have by definition no tensile stresses, we have also considered here isostatic structures with unrestricted stresses. Our results suggest the following picture: on disordered isostatic networks the distribution $P(G)$ of Green functions is a truncated power law, with a cut-off that grows exponentially with distance. If no sign constraint exists for stresses, the stress distribution $P(\lambda)$ presents surprising characteristics: all moments of the stress distribution of an order larger than one grow *exponentially* with depth. In other words, the distribution of stresses is exponentially broad, but its first moment only grows linearly with depth. This is a consequence of cancellations among stresses of different signs, and is required by weight conservation.

When the external pressure is increased, excess contacts appear, the network is no longer isostatic but hyperstatic. Hyperstaticity produces a bounded distribution of Green functions, and now stress distributions decay in a quasi-Gaussian fashion for large stresses. This transition from 'anomalous' (exponential tails in $P(\lambda)$) to 'normal' (Gaussian tails) elastic behaviour under increasing pressure has been observed in recent experiments [6] and numerical simulations [30], in which the packing fraction γ is controlled. The mechanism by which this transition happens is, as argued here, that excess contacts damp the effect of multiplicative noise by introducing an extra term in the equations which rule the propagation of Green functions. This extra term is due to the fact that an overconstrained network opposes deformation, requiring a finite expenditure of elastic energy to be deformed. A discussion of specific models describing this 'noise-induced' transition is left for a forthcoming publication [25].

Acknowledgment

Financial support from CONACYT México is acknowledged.

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